Heat Transfer Mechanisms

- **Heat**: is the form of energy that can be transferred from one system to another as a result of temperature difference.
- **Heat Transfer**: The science that deals with the determination of the *rates of such energy transfers.*

Types of Heat Transfer

- Conduction
- Convection
- Radiation

Conduction

- **Conduction** is the transfer of energy from the more energetic particles of a substance to the adjacent less energetic ones as a result of interactions between the particles.
- Conduction can take place in solids, liquids, or gases.
- In **gases and liquids**, conduction is due to the *collisions and diffusion of the* molecules during their random motion.
- In **solids**, it is due to the combination of *vibrations of the molecules and the energy transport by free electrons.*

Rate of Heat Conduction

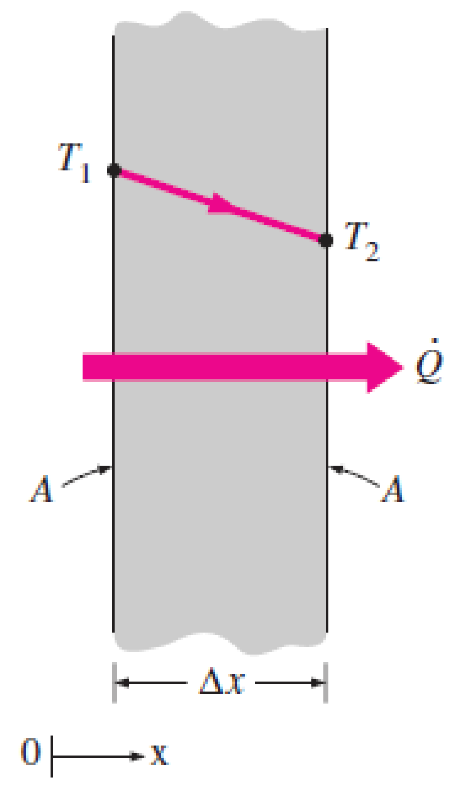
$$\text{Rate of heat conduction} \propto \frac{(\text{Area})(\text{Temperature difference})}{\text{Thickness}}$$

or,

$$\dot{Q}_{\text{cond}} = kA\,\frac{T_1 - T_2}{\Delta x} = -kA\,\frac{\Delta T}{\Delta x} \qquad \text{(W)}$$

where the constant of proportionality **k** represents the thermal conductivity of the material, which is a measure of the ability of a material to conduct heat.

$$\dot{Q}_{\text{cond}} = -kA\,\frac{dT}{dx} \qquad \text{(W)}$$



Here, **A** represents area *NORMAL* to the direction of heat trasnfer

EXAMPLE 1–5 The Cost of Heat Loss through a Roof

The roof of an electrically heated home is 6 m long, 8 m wide, and 0.25 m thick, and is made of a flat layer of concrete whose thermal conductivity is $k = 0.8$ W/m · °C (Fig. 1–24). The temperatures of the inner and the outer surfaces of the roof one night are measured to be 15°C and 4°C, respectively, for a period of 10 hours. Determine (*a*) the rate of heat loss through the roof that night and (*b*) the cost of that heat loss to the home owner if the cost of electricity is \$0.08/kWh.

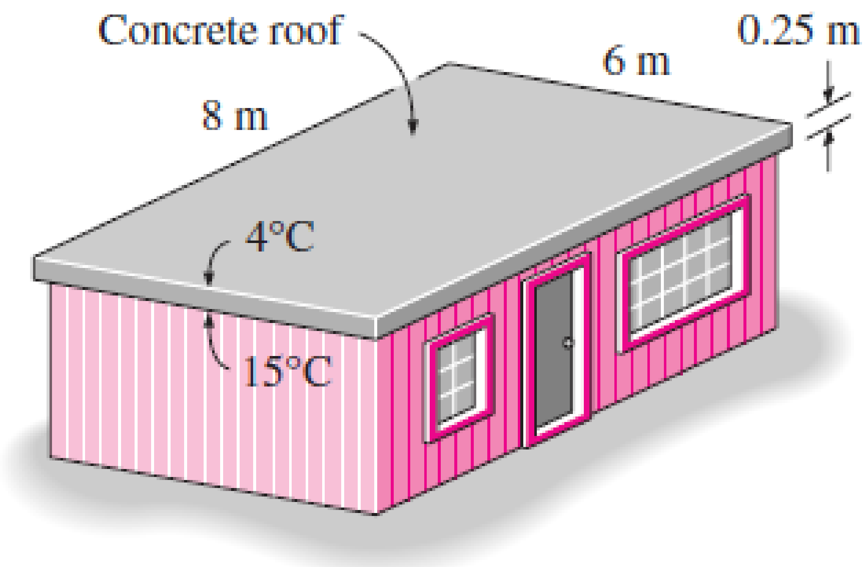


FIGURE 1–24
Schematic for Example 1–5.

SOLUTION The inner and outer surfaces of the flat concrete roof of an electrically heated home are maintained at specified temperatures during a night. The heat loss through the roof and its cost that night are to be determined.

Assumptions **1** Steady operating conditions exist during the entire night since the surface temperatures of the roof remain constant at the specified values. **2** Constant properties can be used for the roof.

Properties The thermal conductivity of the roof is given to be $k = 0.8$ W/m · °C.

Analysis (*a*) Noting that heat transfer through the roof is by conduction and the area of the roof is $A = 6 \text{ m} \times 8 \text{ m} = 48 \text{ m}^2$, the steady rate of heat transfer through the roof is determined to be

$$\dot{Q} = kA\frac{T_1 - T_2}{L} = (0.8 \text{ W/m} \cdot {}^\circ\text{C})(48 \text{ m}^2)\frac{(15 - 4)^\circ\text{C}}{0.25 \text{ m}} = \mathbf{1690 \text{ W} = 1.69 \text{ kW}}$$

(*b*) The amount of heat lost through the roof during a 10-hour period and its cost are determined from

$$Q = \dot{Q}\,\Delta t = (1.69 \text{ kW})(10 \text{ h}) = 16.9 \text{ kWh}$$

$$\text{Cost} = (\text{Amount of energy})(\text{Unit cost of energy})$$

$$= (16.9 \text{ kWh})(\$0.08/\text{kWh}) = \mathbf{\$1.35}$$

Discussion The cost to the home owner of the heat loss through the roof that night was \$1.35. The total heating bill of the house will be much larger since the heat losses through the walls are not considered in these calculations.

Thermal Conductivity

- The **thermal conductivity** of a material can be defined as *the rate of heat transfer through a unit thickness of the material per unit area per unit temperature difference.*

TABLE 1–1

The thermal conductivities of some materials at room temperature

Material	k, W/m · °C*
Diamond	2300
Silver	429
Copper	401
Gold	317
Aluminum	237
Iron	80.2
Mercury (l)	8.54
Glass	0.78
Brick	0.72
Water (l)	0.613
Human skin	0.37
Wood (oak)	0.17
Helium (g)	0.152
Soft rubber	0.13
Glass fiber	0.043
Air (g)	0.026
Urethane, rigid foam	0.026

*Multiply by 0.5778 to convert to Btu/h · ft · °F.

TABLE 1–2

The thermal conductivity of an alloy is usually much lower than the thermal conductivity of either metal of which it is composed

Pure metal or alloy	*k*, W/m · °C, at 300 K
Copper	401
Nickel	91
Constantan (55% Cu, 45% Ni)	23
Copper	401
Aluminum	237
Commercial bronze (90% Cu, 10% Al)	52

TABLE 1–3

Thermal conductivities of materials vary with temperature

T, K	Copper	Aluminum
100	482	302
200	413	237
300	401	237
400	393	240
600	379	231
800	366	218

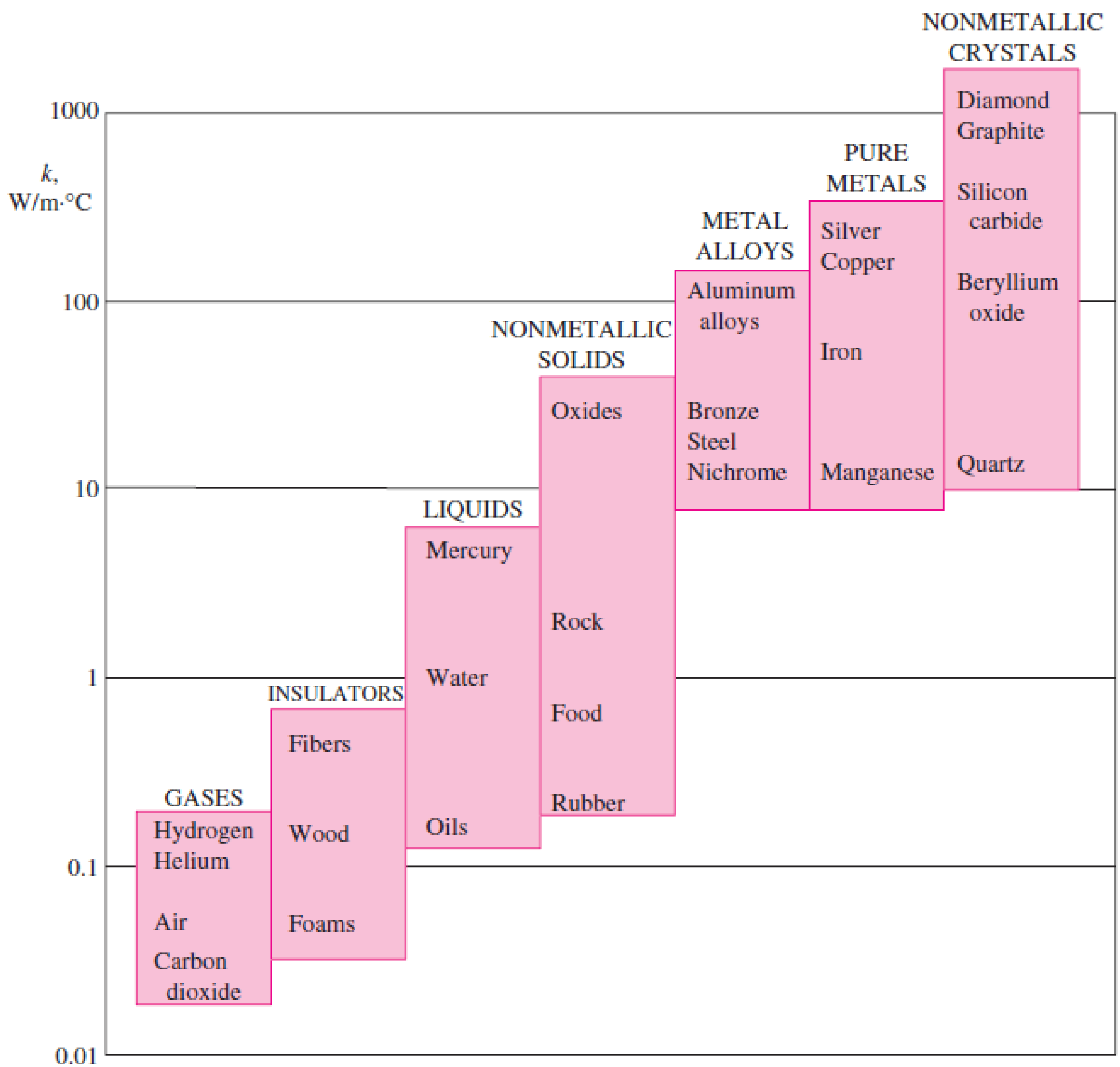


FIGURE 1–26

The range of thermal conductivity of various materials at room temperature.

Thermal Diffusivity

- The thermal diffusivity represents how fast heat diffuses through a material and is defined as

$$\alpha = \frac{\text{Heat conducted}}{\text{Heat stored}} = \frac{k}{\rho C_p} \qquad (\mathrm{m^2/s})$$

The product ρC_p represents heat capacity

EXAMPLE 1–6 **Measuring the Thermal Conductivity of a Material**

A common way of measuring the thermal conductivity of a material is to sandwich an electric thermofoil heater between two identical samples of the material, as shown in Fig. 1–29. The thickness of the resistance heater, including its cover, which is made of thin silicon rubber, is usually less than 0.5 mm. A circulating fluid such as tap water keeps the exposed ends of the samples at constant temperature. The lateral surfaces of the samples are well insulated to ensure that heat transfer through the samples is one-dimensional. Two thermocouples are embedded into each sample some distance L apart, and a differential thermometer reads the temperature drop ΔT across this distance along each sample. When steady operating conditions are reached, the total rate of heat transfer through both samples becomes equal to the electric power drawn by the heater, which is determined by multiplying the electric current by the voltage.

In a certain experiment, cylindrical samples of diameter 5 cm and length 10 cm are used. The two thermocouples in each sample are placed 3 cm apart. After initial transients, the electric heater is observed to draw 0.4 A at 110 V, and both differential thermometers read a temperature difference of 15°C. Determine the thermal conductivity of the sample.

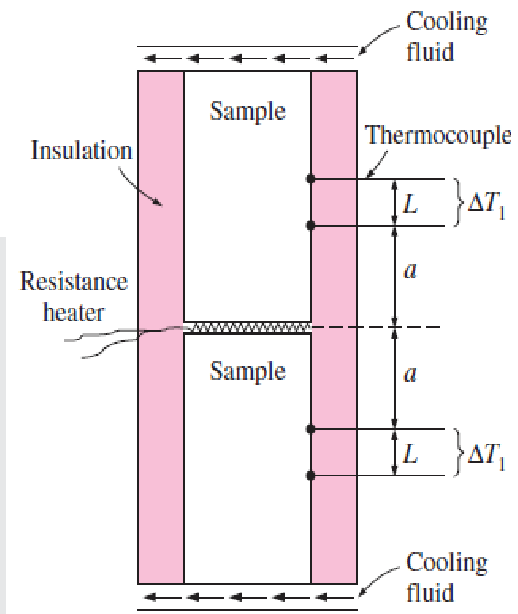


FIGURE 1–29

Apparatus to measure the thermal conductivity of a material using two identical samples and a thin resistance heater (Example 1–6).

SOLUTION The thermal conductivity of a material is to be determined by ensuring one-dimensional heat conduction, and by measuring temperatures when steady operating conditions are reached.

Assumptions **1** Steady operating conditions exist since the temperature readings do not change with time. **2** Heat losses through the lateral surfaces of the apparatus are negligible since those surfaces are well insulated, and thus the entire heat generated by the heater is conducted through the samples. **3** The apparatus possesses thermal symmetry.

Analysis The electrical power consumed by the resistance heater and converted to heat is

$$\dot{W}_e = VI = (110\ \text{V})(0.4\ \text{A}) = 44\ \text{W}$$

The rate of heat flow through each sample is

$$\dot{Q} = \tfrac{1}{2}\dot{W}_e = \tfrac{1}{2} \times (44\ \text{W}) = 22\ \text{W}$$

since only half of the heat generated will flow through each sample because of symmetry. Reading the same temperature difference across the same distance in each sample also confirms that the apparatus possesses thermal symmetry. The heat transfer area is the area normal to the direction of heat flow, which is the cross-sectional area of the cylinder in this case:

$$A = \tfrac{1}{4}\pi D^2 = \tfrac{1}{4}\pi(0.05\ \text{m})^2 = 0.00196\ \text{m}^2$$

Noting that the temperature drops by 15°C within 3 cm in the direction of heat flow, the thermal conductivity of the sample is determined to be

$$\dot{Q} = kA\frac{\Delta T}{L} \quad \rightarrow \quad k = \frac{\dot{Q}L}{A\,\Delta T} = \frac{(22\ \text{W})(0.03\ \text{m})}{(0.00196\ \text{m}^2)(15°\text{C})} = \mathbf{22.4\ W/m \cdot °C}$$

Discussion Perhaps you are wondering if we really need to use two samples in the apparatus, since the measurements on the second sample do not give any additional information. It seems like we can replace the second sample by insulation. Indeed, we do not need the second sample; however, it enables us to verify the temperature measurements on the first sample and provides thermal symmetry, which reduces experimental error.

Convection

- **Convection** is the mode of energy transfer between a solid surface and the adjacent liquid or gas that is in motion, and it involves the combined effects of c*onduction and fluid motion.*

- The faster the fluid motion, the greater the convection heat transfer.

- In the absence of any bulk fluid motion, heat transfer between a solid surface and the adjacent fluid is by pure conduction.

- The presence of bulk motion of the fluid enhances the heat transfer between the solid surface and the fluid.

- **Forced convection** if the fluid is forced to flow over the surface by external means such as a fan, pump, or the wind.
- **Natural (or free) convection** if the fluid motion is caused by buoyancy forces that are induced by density differences due to the variation of temperature in the fluid (Fig. 1–32).

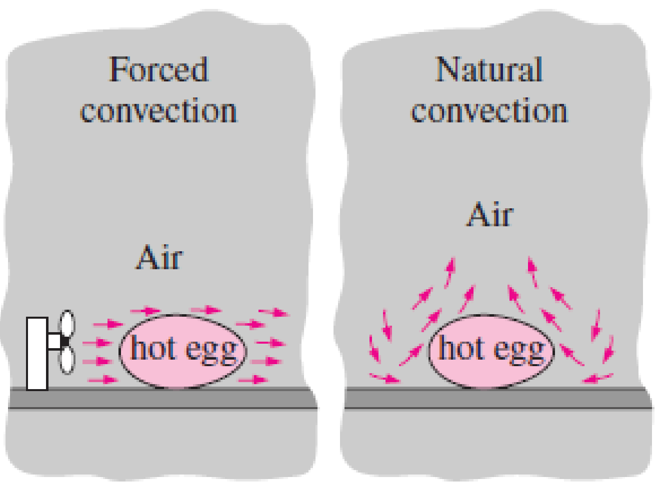


FIGURE 1–32
The cooling of a boiled egg by forced and natural convection.

- Despite the complexity of convection, the rate of *convection heat transfer is* observed to be proportional to the temperature difference, and is conveniently expressed by **Newton's law of cooling as**

$$\dot{Q}_{\text{conv}} = hA_s(T_s - T_\infty) \qquad \text{(W)} \qquad \text{(1-24)}$$

where h is the *convection heat transfer coefficient* in $\text{W/m}^2 \cdot {}^\circ\text{C}$ or $\text{Btu/h} \cdot \text{ft}^2 \cdot {}^\circ\text{F}$, A_s is the surface area through which convection heat transfer takes place, T_s is the surface temperature, and T_∞ is the temperature of the fluid sufficiently far from the surface. Note that at the surface, the fluid temperature equals the surface temperature of the solid.

TABLE 1–5

Typical values of convection heat transfer coefficient

Type of convection	h, W/m^2 · °C*
Free convection of gases	2–25
Free convection of liquids	10–1000
Forced convection of gases	25–250
Forced convection of liquids	50–20,000
Boiling and condensation	2500–100,000

*Multiply by 0.176 to convert to Btu/h · ft^2 · °F.

EXAMPLE 1–8 Measuring Convection Heat Transfer Coefficient

A 2-m-long, 0.3-cm-diameter electrical wire extends across a room at 15°C, as shown in Fig. 1–33. Heat is generated in the wire as a result of resistance heating, and the surface temperature of the wire is measured to be 152°C in steady operation. Also, the voltage drop and electric current through the wire are measured to be 60 V and 1.5 A, respectively. Disregarding any heat transfer by radiation, determine the convection heat transfer coefficient for heat transfer between the outer surface of the wire and the air in the room.

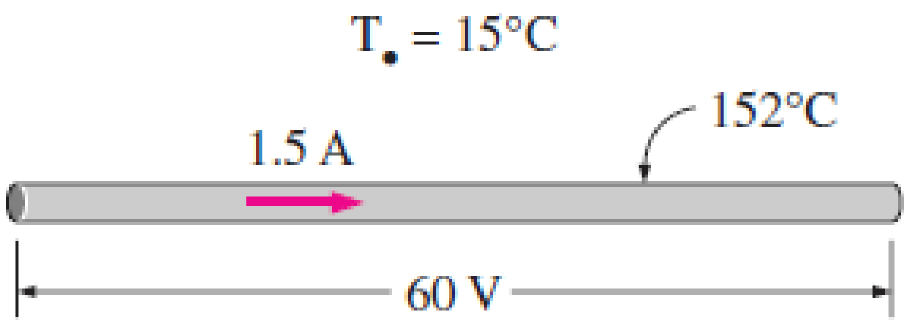


FIGURE 1–33
Schematic for Example 1–8.

SOLUTION The convection heat transfer coefficient for heat transfer from an electrically heated wire to air is to be determined by measuring temperatures when steady operating conditions are reached and the electric power consumed.

Assumptions **1** Steady operating conditions exist since the temperature readings do not change with time. **2** Radiation heat transfer is negligible.

Analysis When steady operating conditions are reached, the rate of heat loss from the wire will equal the rate of heat generation in the wire as a result of resistance heating. That is,

$$\dot{Q} = \dot{E}_{\text{generated}} = VI = (60 \text{ V})(1.5 \text{ A}) = 90 \text{ W}$$

The surface area of the wire is

$$A_s = \pi DL = \pi(0.003 \text{ m})(2 \text{ m}) = 0.01885 \text{ m}^2$$

Newton's law of cooling for convection heat transfer is expressed as

$$\dot{Q}_{\text{conv}} = hA_s (T_s - T_\infty)$$

Disregarding any heat transfer by radiation and thus assuming all the heat loss from the wire to occur by convection, the convection heat transfer coefficient is determined to be

$$h = \frac{\dot{Q}_{\text{conv}}}{A_s(T_s - T_\infty)} = \frac{90 \text{ W}}{(0.01885 \text{ m}^2)(152 - 15)°\text{C}} = \mathbf{34.9 \text{ W/m}^2 \cdot °\text{C}}$$

Discussion Note that the simple setup described above can be used to determine the average heat transfer coefficients from a variety of surfaces in air. Also, heat transfer by radiation can be eliminated by keeping the surrounding surfaces at the temperature of the wire.

Radiation

- **Radiation** is the energy emitted by matter in the form of *electromagnetic waves (or photons) as a result of the changes in the electronic configurations* of the atoms or molecules.
- Unlike conduction and convection, the transfer of energy by radiation does not require the presence of an *intervening medium.*
- In fact, energy transfer by radiation is fastest (at the speed of light) and it suffers no attenuation in a vacuum. This is how the energy of the sun reaches the earth.

- In heat transfer studies we are interested in thermal radiation, which is the form of radiation emitted by bodies because of their temperature.
- It differs from other forms of electromagnetic radiation such as x-rays, gamma rays, microwaves, radio waves, and television waves that are not related to temperature.
- All bodies at a temperature above absolute zero emit thermal radiation.

The maximum rate of radiation that can be emitted from a surface at an absolute temperature T_s (in K or R) is given by the **Stefan–Boltzmann law** as

$$\dot{Q}_{\text{emit, max}} = \sigma A_s T_s^4 \qquad \text{(W)} \tag{1-25}$$

where $\sigma = 5.67 \times 10^{-8}\ \text{W/m}^2 \cdot \text{K}^4$ or $0.1714 \times 10^{-8}\ \text{Btu/h} \cdot \text{ft}^2 \cdot \text{R}^4$ is the *Stefan–Boltzmann constant.* The idealized surface that emits radiation at this maximum rate is called a **blackbody,** and the radiation emitted by a blackbody is called **blackbody radiation** (Fig. 1–34). The radiation emitted by all real surfaces is less than the radiation emitted by a blackbody at the same temperature, and is expressed as

$$\dot{Q}_{\text{emit}} = \varepsilon \sigma A_s T_s^4 \qquad \text{(W)} \tag{1-26}$$

where ε is the **emissivity** of the surface. The property emissivity, whose value is in the range $0 \leq \varepsilon \leq 1$, is a measure of how closely a surface approximates a blackbody for which $\varepsilon = 1$. The emissivities of some surfaces are given in Table 1–6.

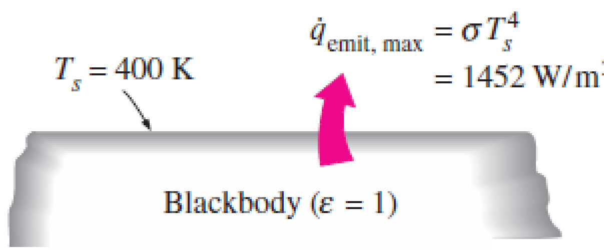


FIGURE 1–34
Blackbody radiation represents the *maximum amount of radiation that can be emitted from a surface at a specified temperature.*

TABLE 1–6

Emissivities of some materials at 300 K

Material	Emissivity
Aluminum foil	0.07
Anodized aluminum	0.82
Polished copper	0.03
Polished gold	0.03
Polished silver	0.02
Polished stainless steel	0.17
Black paint	0.98
White paint	0.90
White paper	0.92–0.97
Asphalt pavement	0.85–0.93
Red brick	0.93–0.96
Human skin	0.95
Wood	0.82–0.92
Soil	0.93–0.96
Water	0.96
Vegetation	0.92–0.96

Another important radiation property of a surface is its **absorptivity** α, which is the fraction of the radiation energy incident on a surface that is absorbed by the surface. Like emissivity, its value is in the range $0 \leq \alpha \leq 1$. A blackbody absorbs the entire radiation incident on it. That is, a blackbody is a perfect absorber ($\alpha = 1$) as it is a perfect emitter.

In general, both ε and α of a surface depend on the temperature and the wavelength of the radiation. **Kirchhoff's law** of radiation states that the emissivity and the absorptivity of a surface at a given temperature and wavelength are equal. In many practical applications, the surface temperature and the temperature of the source of incident radiation are of the same order of magnitude, and the average absorptivity of a surface is taken to be equal to its average emissivity. The rate at which a surface absorbs radiation is determined from (Fig. 1–35)

$$\dot{Q}_{\text{absorbed}} = \alpha \dot{Q}_{\text{incident}} \qquad \text{(W)} \tag{1-27}$$

where $\dot{Q}_{\text{incident}}$ is the rate at which radiation is incident on the surface and α is the absorptivity of the surface. For opaque (nontransparent) surfaces, the portion of incident radiation not absorbed by the surface is reflected back.

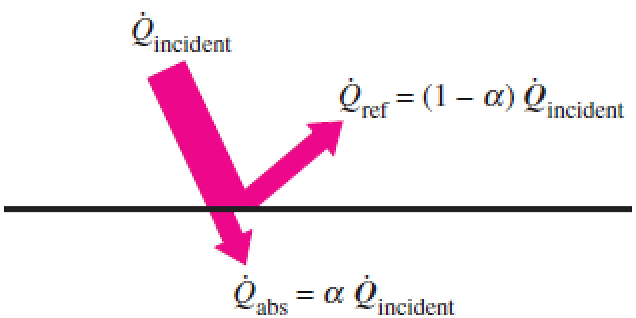


FIGURE 1–35

The absorption of radiation incident on an opaque surface of absorptivity α.

When a surface of emissivity ε and surface area A_s at an *absolute temperature* T_s is *completely enclosed* by a much larger (or black) surface at absolute temperature T_{surr} separated by a gas (such as air) that does not intervene with radiation, the net rate of radiation heat transfer between these two surfaces is given by (Fig. 1–36)

$$\dot{Q}_{rad} = \varepsilon\sigma A_s (T_s^4 - T_{surr}^4) \qquad \text{(W)} \tag{1-28}$$

Then the *total* heat transfer rate to or from a surface by convection and radiation is expressed as

$$\dot{Q}_{total} = h_{combined} A_s (T_s - T_\infty) \qquad \text{(W)}$$

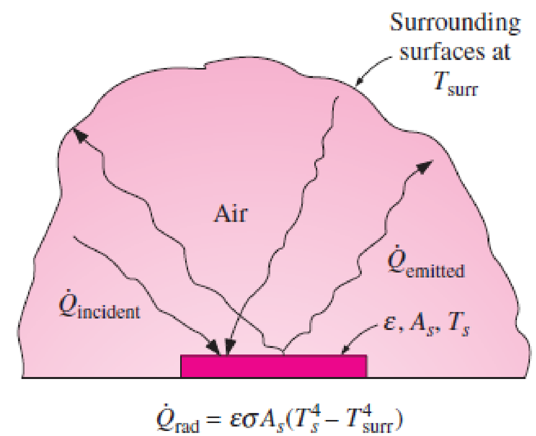


FIGURE 1–36
Radiation heat transfer between a surface and the surfaces surrounding it.

Radiation is usually significant relative to conduction or natural convection, but negligible relative to forced convection. Thus radiation in forced convection applications is usually disregarded, especially when the surfaces involved have low emissivities and low to moderate temperatures.

EXAMPLE 1–9 Radiation Effect on Thermal Comfort

It is a common experience to feel "chilly" in winter and "warm" in summer in our homes even when the thermostat setting is kept the same. This is due to the so called "radiation effect" resulting from radiation heat exchange between our bodies and the surrounding surfaces of the walls and the ceiling.

Consider a person standing in a room maintained at 22°C at all times. The inner surfaces of the walls, floors, and the ceiling of the house are observed to be at an average temperature of 10°C in winter and 25°C in summer. Determine the rate of radiation heat transfer between this person and the surrounding surfaces if the exposed surface area and the average outer surface temperature of the person are 1.4 m^2 and 30°C, respectively (Fig. 1–37).

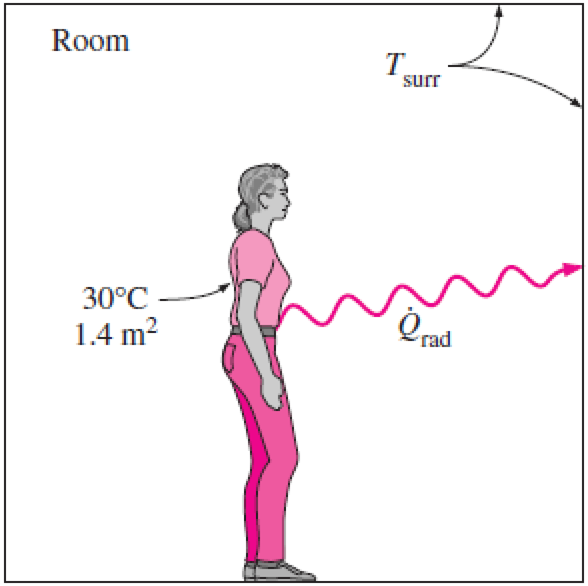


FIGURE 1–37
Schematic for Example 1–9.

SOLUTION The rates of radiation heat transfer between a person and the surrounding surfaces at specified temperatures are to be determined in summer and winter.

Assumptions **1** Steady operating conditions exist. **2** Heat transfer by convection is not considered. **3** The person is completely surrounded by the interior surfaces of the room. **4** The surrounding surfaces are at a uniform temperature.

Properties The emissivity of a person is $\varepsilon = 0.95$ (Table 1–6).

Analysis The net rates of radiation heat transfer from the body to the surrounding walls, ceiling, and floor in winter and summer are

$$\begin{aligned}\dot{Q}_{\text{rad, winter}} &= \varepsilon\sigma A_s\,(T_s^4 - T_{\text{surr, winter}}^4)\\ &= (0.95)(5.67\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4)(1.4\ \text{m}^2)\\ &\quad\times[(30+273)^4-(10+273)^4]\ \text{K}^4\\ &= \mathbf{152\ W}\end{aligned}$$

and

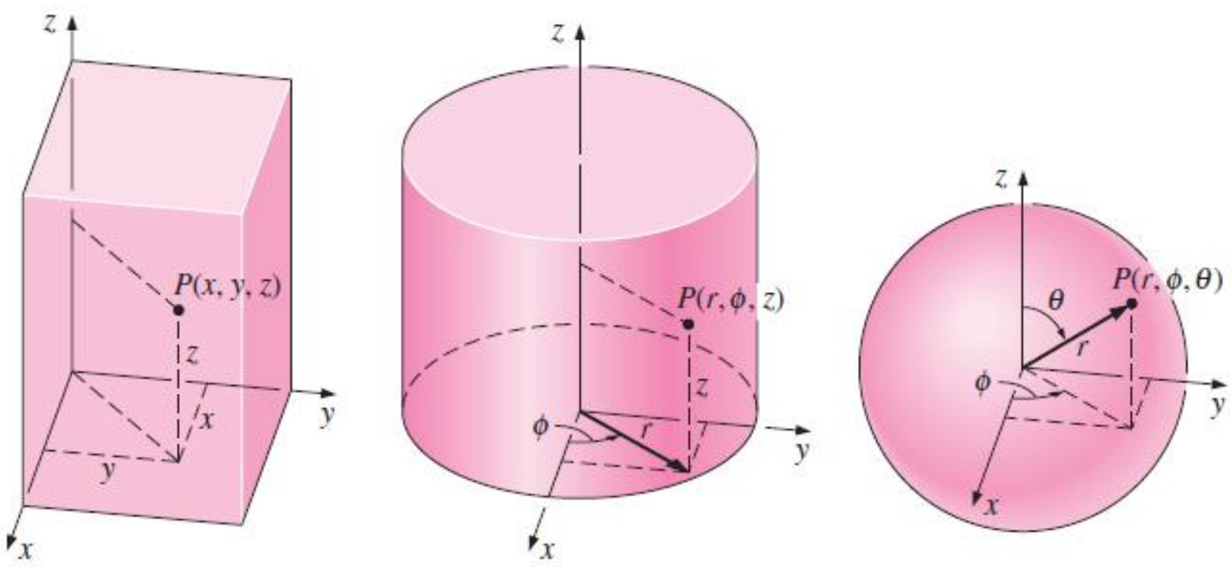
$$\begin{aligned}\dot{Q}_{\text{rad, summer}} &= \varepsilon\sigma A_s\,(T_s^4 - T_{\text{surr, summer}}^4)\\ &= (0.95)(5.67\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4)(1.4\ \text{m}^2)\\ &\quad\times[(30+273)^4-(25+273)^4]\ \text{K}^4\\ &= \mathbf{40.9\ W}\end{aligned}$$

Discussion Note that we must use *absolute temperatures* in radiation calculations. Also note that the rate of heat loss from the person by radiation is almost four times as large in winter than it is in summer, which explains the "chill" we feel in winter even if the thermostat setting is kept the same.

Heat Conduction Equation

- Heat transfer has direction as well as magnitude. The rate of heat conduction in a specified direction is proportional to the temperature gradient, which is the change in temperature per unit length in that direction.
- Heat transfer is a vector quantity.
- The specification of the temperature at a point in a medium first requires the specification of the location of that point.
- This can be done by choosing a suitable coordinate system such as the rectangular, cylindrical, or spherical coordinates, depending on the geometry involved, and a convenient reference point (the origin).

- The *location of a point is specified as (x, y, z) in rectangular coordinates, as* (*r, , z) in cylindrical coordinates, and as (r, ,) in spherical coordinates,* where the distances *x, y, z, and r and the angles and are as shown in Figure*2–3.



(*a*) Rectangular coordinates (*b*) Cylindrical coordinates (*c*) Spherical coordinates

FIGURE 2–3
The various distances and angles involved when describing the location of a point in different coordinate systems.

Steady V/s Transient

- The term *steady implies no change* with time at any point within the medium, while *transient implies variation with time or time dependence.*

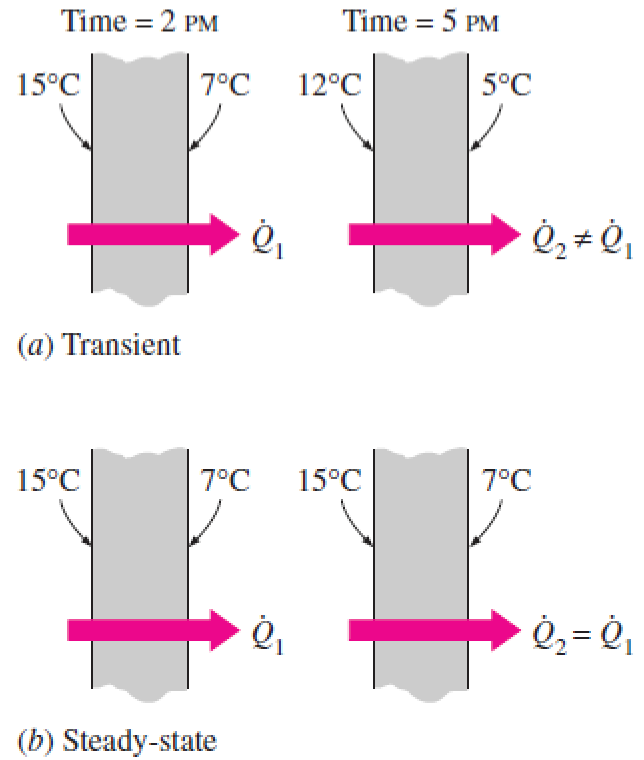


FIGURE 2–4
Steady and transient heat conduction in a plane wall.

- In the special case of variation with time but not with position, the temperature of the medium changes uniformly with time and such heat transfer systems are called lumped systems.
- A small metal object such as a thermocouple junction or a thin copper wire, for example, can be analyzed as a lumped system during a heating or cooling process.

Multidimensional Heat Transfer

- Heat transfer problems are also classified as being one-dimensional, two-dimensional, or three-dimensional, depending on the relative magnitudes of heat transfer rates in different directions and the level of accuracy desired.
- Mostly heat transfer is three-dimensional.

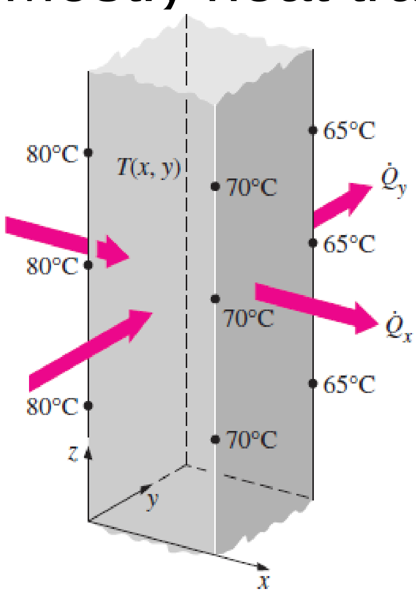


FIGURE 2–5
Two-dimensional heat transfer in a long rectangular bar.

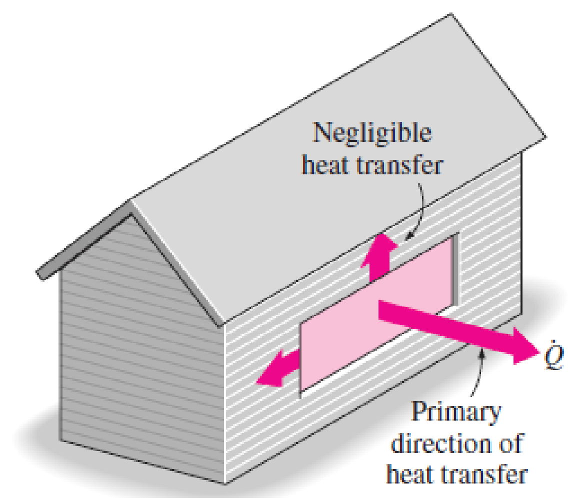


FIGURE 2–6
Heat transfer through the window of a house can be taken to be one-dimensional.

- **Fourier's law of heat conduction** for one-dimensional heat conduction is defined as

$$\dot{Q}_{\text{cond}} = -kA\frac{dT}{dx} \qquad (\text{W}) \qquad \textbf{(2-1)}$$

where k is the *thermal conductivity* of the material, which is a measure of the ability of a material to conduct heat, and dT/dx is the *temperature gradient*, which is the slope of the temperature curve on a T-x diagram (Fig. 2–7).

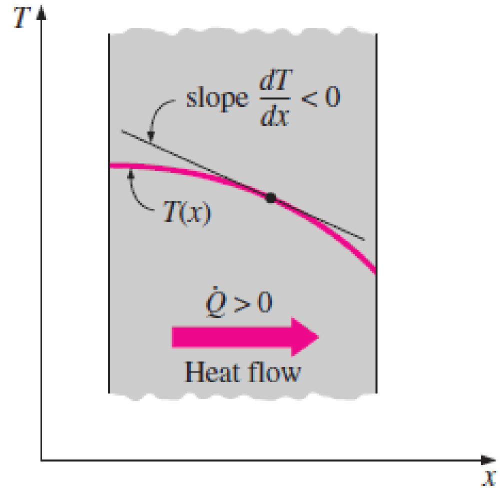


FIGURE 2–7
The temperature gradient dT/dx is simply the slope of the temperature curve on a T-x diagram.

- General formulation for Fourier's law can be obtained as shown below

If n is the normal of the isothermal surface at point P, the rate of heat conduction at that point can be expressed by Fourier's law as

$$\dot{Q}_n = -kA\frac{\partial T}{\partial n} \qquad \text{(W)} \tag{2-2}$$

In rectangular coordinates, the heat conduction vector can be expressed in terms of its components as

$$\vec{\dot{Q}}_n = \dot{Q}_x\vec{i} + \dot{Q}_y\vec{j} + \dot{Q}_z\vec{k} \tag{2-3}$$

where $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the unit vectors, and $\dot{Q}_x$, $\dot{Q}_y$, and $\dot{Q}_z$ are the magnitudes of the heat transfer rates in the x-, y-, and z-directions, which again can be determined from Fourier's law as

$$\dot{Q}_x = -kA_x\frac{\partial T}{\partial x}, \qquad \dot{Q}_y = -kA_y\frac{\partial T}{\partial y}, \qquad \text{and} \qquad \dot{Q}_z = -kA_z\frac{\partial T}{\partial z} \tag{2-4}$$

Here A_x, A_y and A_z are heat conduction areas normal to the x-, y-, and z-directions, respectively (Fig. 2–8).

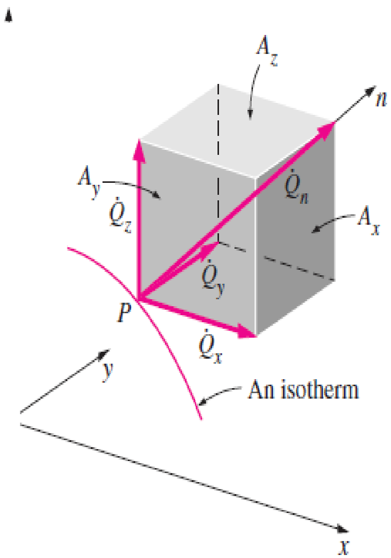


FIGURE 2–8
The heat transfer vector is always normal to an isothermal rface and can be resolved into its omponents like any other vector.

Heat Generation

- Heat generation is a volumetric phenomenon.
- It occurs throughout the body of a medium.
- The rate of heat generation in a medium is usually specified per unit volume and is denoted by $\dot{g}$, whose unit is W/m^3

$$\dot{G} = \int_V \dot{g} dV \qquad \text{(W)} \qquad \text{(2-5)}$$

In the special case of *uniform* heat generation, as in the case of electric resistance heating throughout a homogeneous material, the relation in Eq. 2–5 reduces to $\dot{G} = \dot{g}V$, where $\dot{g}$ is the constant rate of heat generation per unit volume.

EXAMPLE 2–2 Heat Generation in a Hair Dryer

The resistance wire of a 1200-W hair dryer is 80 cm long and has a diameter of $D = 0.3$ cm (Fig. 2–12). Determine the rate of heat generation in the wire per unit volume, in W/cm^3, and the heat flux on the outer surface of the wire as a result of this heat generation.

FIGURE 2–12
Schematic for Example 2–2.

SOLUTION The power consumed by the resistance wire of a hair dryer is given. The heat generation and the heat flux are to be determined.

Assumptions Heat is generated uniformly in the resistance wire.

Analysis A 1200-W hair dryer will convert electrical energy into heat in the wire at a rate of 1200 W. Therefore, the rate of heat generation in a resistance wire is equal to the power consumption of a resistance heater. Then the rate of heat generation in the wire per unit volume is determined by dividing the total rate of heat generation by the volume of the wire,

$$\dot{g} = \frac{\dot{G}}{V_{\text{wire}}} = \frac{\dot{G}}{(\pi D^2/4)L} = \frac{1200\ \text{W}}{[\pi(0.3\ \text{cm})^2/4](80\ \text{cm})} = \mathbf{212\ W/cm^3}$$

Similarly, heat flux on the outer surface of the wire as a result of this heat generation is determined by dividing the total rate of heat generation by the surface area of the wire,

$$\dot{q} = \frac{\dot{G}}{A_{\text{wire}}} = \frac{\dot{G}}{\pi DL} = \frac{1200\ \text{W}}{\pi(0.3\ \text{cm})(80\ \text{cm})} = \mathbf{15.9\ W/cm^2}$$

Heat Conduction in Large Plane Wall

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x+\Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of the}\\ \text{element}\end{pmatrix}$$

or

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} \quad \text{(2-6)}$$

But the change in the energy content of the element and the rate of heat generation within the element can be expressed as

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mC(T_{t+\Delta t} - T_t) = \rho C A \Delta x (T_{t+\Delta t} - T_t) \quad \text{(2-7)}$$

$$\dot{G}_{\text{element}} = \dot{g} V_{\text{element}} = \dot{g} A \Delta x \quad \text{(2-8)}$$

Substituting into Equation 2–6, we get

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{g} A \Delta x = \rho C A \Delta x \frac{T_{t+\Delta t} - T_t}{\Delta t} \quad \text{(2-9)}$$

Dividing by $A\Delta x$ gives

$$-\frac{1}{A}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} + \dot{g} = \rho C \frac{T_{t+\Delta t} - T_t}{\Delta t} \quad \text{(2-10)}$$

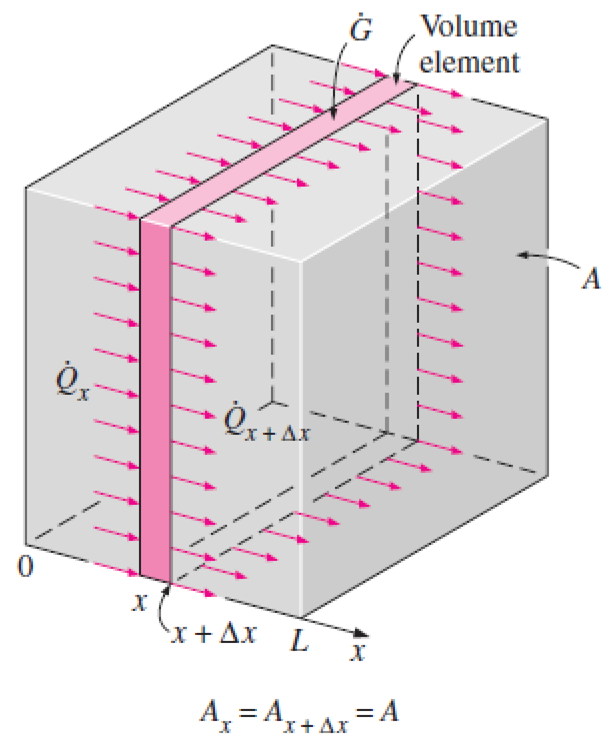


FIGURE 2–13
One-dimensional heat conduction through a volume element in a large plane wall.

Taking the limit as $\Delta x \to 0$ and $\Delta t \to 0$ yields

$$\frac{1}{A}\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t} \tag{2-11}$$

since, from the definition of the derivative and Fourier's law of heat conduction,

$$\lim_{\Delta x \to 0}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} = \frac{\partial \dot{Q}}{\partial x} = \frac{\partial}{\partial x}\left(-kA\frac{\partial T}{\partial x}\right) \tag{2-12}$$

Noting that the area A is constant for a plane wall, the one-dimensional transient heat conduction equation in a plane wall becomes

Variable conductivity:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t} \tag{2-13}$$

The thermal conductivity k of a material, in general, depends on the temperature T (and therefore x), and thus it cannot be taken out of the derivative. However, the *thermal conductivity* in most practical applications can be assumed to remain *constant* at some average value. The equation above in that case reduces to

Constant conductivity:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2-14}$$

where the property $\alpha = k/\rho C$ is the **thermal diffusivity** of the material and represents how fast heat propagates through a material. It reduces to the following forms under specified conditions (Fig. 2–14):

(1) *Steady-state:* $(\partial/\partial t = 0)$

$$\frac{d^2T}{dx^2} + \frac{\dot{g}}{k} = 0 \tag{2-15}$$

(2) *Transient, no heat generation:* $(\dot{g} = 0)$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2-16}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t = 0$ and $\dot{g} = 0)$

$$\frac{d^2T}{dx^2} = 0 \tag{2-17}$$

Heat Conduction in a Long Cylinder

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } r\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } r+\Delta r\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of the}\\ \text{element}\end{pmatrix}$$

or

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} \qquad \text{(2-18}$$

The change in the energy content of the element and the rate of heat generation within the element can be expressed as

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mC(T_{t+\Delta t} - T_t) = \rho C A \Delta r (T_{t+\Delta t} - T_t) \qquad \text{(2-19}$$

$$\dot{G}_{\text{element}} = \dot{g} V_{\text{element}} = \dot{g} A \Delta r \qquad \text{(2-20}$$

Substituting into Eq. 2–18, we get

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{g} A \Delta r = \rho C A \Delta r \frac{T_{t+\Delta t} - T_t}{\Delta t} \qquad \text{(2-21}$$

where $A = 2\pi r L$. You may be tempted to express the area at the *middle* of the element using the *average* radius as $A = 2\pi(r + \Delta r/2)L$. But there is nothing we can gain from this complication since later in the analysis we will take the limit as $\Delta r \to 0$ and thus the term $\Delta r/2$ will drop out. Now dividing the equation above by $A\Delta r$ gives

$$-\frac{1}{A}\frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} + \dot{g} = \rho C \frac{T_{t+\Delta t} - T_t}{\Delta t} \qquad \text{(2-22)}$$

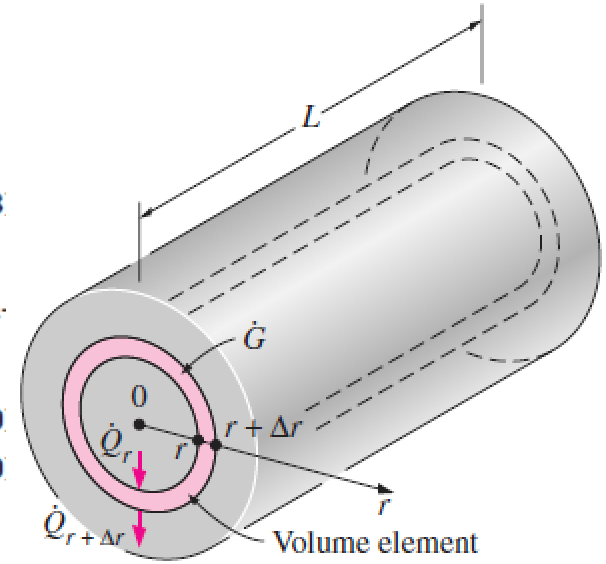


FIGURE 2–15
One-dimensional heat conduction through a volume element in a long cylinder.

Taking the limit as $\Delta r \to 0$ and $\Delta t \to 0$ yields

$$\frac{1}{A}\frac{\partial}{\partial r}\left(kA\frac{\partial T}{\partial r}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t} \tag{2-23}$$

since, from the definition of the derivative and Fourier's law of heat conduction,

$$\lim_{\Delta r \to 0} \frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} = \frac{\partial \dot{Q}}{\partial r} = \frac{\partial}{\partial r}\left(-kA\frac{\partial T}{\partial r}\right) \tag{2-24}$$

Noting that the heat transfer area in this case is $A = 2\pi rL$, the one-dimensional transient heat conduction equation in a cylinder becomes

Variable conductivity:
$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t} \tag{2-25}$$

For the case of constant thermal conductivity, the equation above reduces to

Constant conductivity:
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2-26}$$

where again the property $\alpha = k/\rho C$ is the thermal diffusivity of the material. Equation 2–26 reduces to the following forms under specified conditions (Fig. 2–16):

(1) *Steady-state:*
($\partial/\partial t = 0$)
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{g}}{k} = 0 \tag{2-27}$$

(2) *Transient, no heat generation:*
($\dot{g} = 0$)
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2-28}$$

(3) *Steady-state, no heat generation:*
($\partial/\partial t = 0$ and $\dot{g} = 0$)
$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \tag{2-29}$$

Heat Conduction in a Sphere

Variable conductivity: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k\frac{\partial T}{\partial r}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t} \qquad \text{(2-30)}$$

which, in the case of constant thermal conductivity, reduces to

Constant conductivity: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{(2-31)}$$

where again the property $\alpha = k/\rho C$ is the thermal diffusivity of the material. It reduces to the following forms under specified conditions:

(1) *Steady-state:* ($\partial/\partial t = 0$) $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) + \frac{\dot{g}}{k} = 0 \qquad \text{(2-32)}$$

(2) *Transient, no heat generation:* ($\dot{g} = 0$) $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{(2-33)}$$

(3) *Steady-state, no heat generation:* ($\partial/\partial t = 0$ and $\dot{g} = 0$) $$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0 \quad \text{or} \quad r\frac{d^2T}{dr^2} + 2\frac{dT}{dr} = 0 \qquad \text{(2-34)}$$

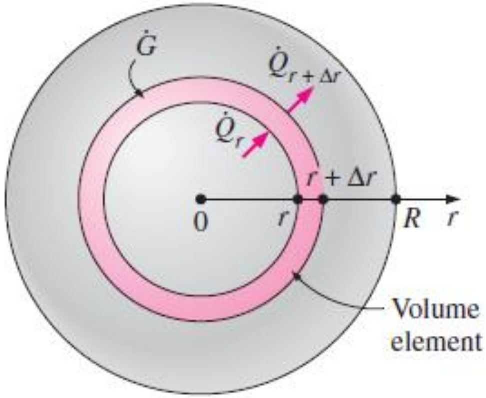


FIGURE 2–17
One-dimensional heat conduction through a volume element in a sphere.

Combine One-Dimensional Equation

- All three cases of one- dimentional can be combined in to one equation which is

$$\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n k \frac{\partial T}{\partial r}\right) + \dot{g} = \rho C \frac{\partial T}{\partial t} \tag{2-35}$$

where $n = 0$ for a plane wall, $n = 1$ for a cylinder, and $n = 2$ for a sphere. In the case of a plane wall, it is customary to replace the variable r by x. This equation can be simplified for steady-state or no heat generation cases as described before.

EXAMPLE 2–4 Heat Conduction in a Resistance Heater

A 2-kW resistance heater wire with thermal conductivity $k = 15$ W/m · °C, diameter $D = 0.4$ cm, and length $L = 50$ cm is used to boil water by immersing it in water (Fig. 2–19). Assuming the variation of the thermal conductivity of the wire with temperature to be negligible, obtain the differential equation that describes the variation of the temperature in the wire during steady operation.

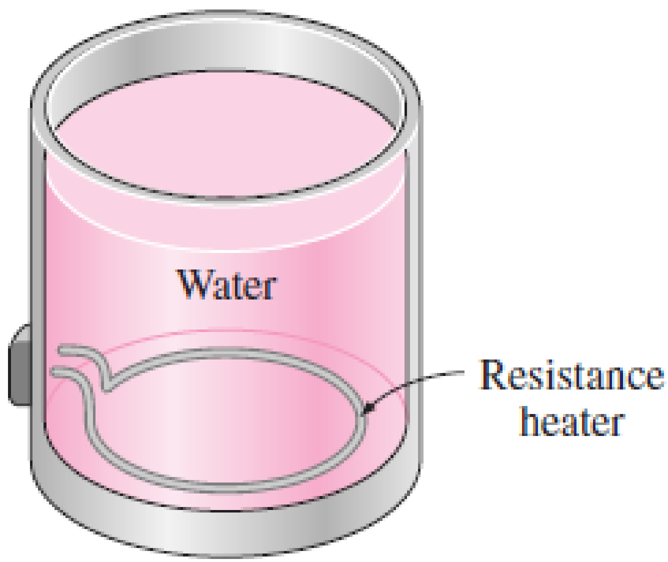


FIGURE 2–19
Schematic for Example 2–4.

SOLUTION The resistance wire can be considered to be a very long cylinder since its length is more than 100 times its diameter. Also, heat is generated uniformly in the wire and the conditions on the outer surface of the wire are uniform. Therefore, it is reasonable to expect the temperature in the wire to vary in the radial r direction only and thus the heat transfer to be one-dimensional. Then we will have $T = T(r)$ during steady operation since the temperature in this case will depend on r only.

The rate of heat generation in the wire per unit volume can be determined from

$$\dot{g} = \frac{\dot{G}}{V_{\text{wire}}} = \frac{\dot{G}}{(\pi D^2/4)L} = \frac{2000 \text{ W}}{[\pi(0.004 \text{ m})^2/4](0.5 \text{ cm})} = 0.318 \times 10^9 \text{ W/m}^3$$

Noting that the thermal conductivity is given to be constant, the differential equation that governs the variation of temperature in the wire is simply Eq. 2–27,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{g}}{k} = 0$$

which is the steady one-dimensional heat conduction equation in cylindrical coordinates for the case of constant thermal conductivity. Note again that the conditions at the surface of the wire have no effect on the differential equation.

EXAMPLE 2–5 Cooling of a Hot Metal Ball in Air

A spherical metal ball of radius R is heated in an oven to a temperature of 600°F throughout and is then taken out of the oven and allowed to cool in ambient air at $T_\infty = 75°F$ by convection and radiation (Fig. 2–20). The thermal conductivity of the ball material is known to vary linearly with temperature. Assuming the ball is cooled uniformly from the entire outer surface, obtain the differential equation that describes the variation of the temperature in the ball during cooling.

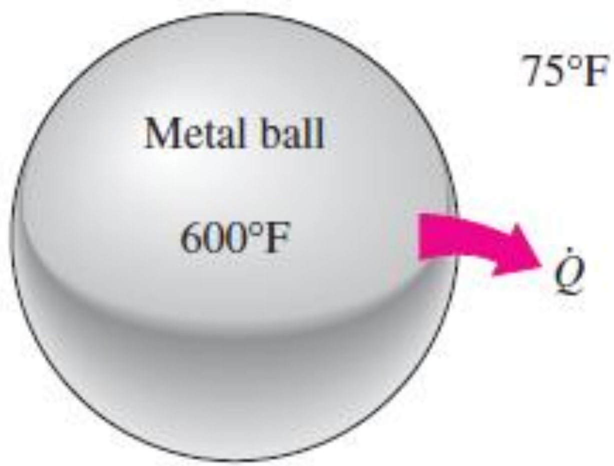


FIGURE 2–20
Schematic for Example 2–5.

SOLUTION The ball is initially at a uniform temperature and is cooled uniformly from the entire outer surface. Also, the temperature at any point in the ball will change with time during cooling. Therefore, this is a one-dimensional transient heat conduction problem since the temperature within the ball will change with the radial distance r and the time t. That is, $T = T(r, t)$.

The thermal conductivity is given to be variable, and there is no heat generation in the ball. Therefore, the differential equation that governs the variation of temperature in the ball in this case is obtained from Eq. 2–30 by setting the heat generation term equal to zero. We obtain

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k \frac{\partial T}{\partial r}\right) = \rho C \frac{\partial T}{\partial t}$$

General Heat Conduction Equations (Rectangular)

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction at}\\ x, y, \text{and } z\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x+\Delta x,\\ y+\Delta y, \text{and } z+\Delta z\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of}\\ \text{the element}\end{pmatrix}$$

or

$$\dot{Q}_x + \dot{Q}_y + \dot{Q}_z - \dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} \quad \textbf{(2-36)}$$

Noting that the volume of the element is $V_{\text{element}} = \Delta x \Delta y \Delta z$, the change in the energy content of the element and the rate of heat generation within the element can be expressed as

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mC(T_{t+\Delta t} - T_t) = \rho C \Delta x \Delta y \Delta z (T_{t+\Delta t} - T_t)$$

$$\dot{G}_{\text{element}} = \dot{g} V_{\text{element}} = \dot{g} \Delta x \Delta y \Delta z$$

Substituting into Eq. 2–36, we get

$$\dot{Q}_x + \dot{Q}_y + \dot{Q}_z - \dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} + \dot{g}\Delta x \Delta y \Delta z = \rho C \Delta x \Delta y \Delta z \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Dividing by $\Delta x \Delta y \Delta z$ gives

$$-\frac{1}{\Delta y \Delta z}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} - \frac{1}{\Delta x \Delta z}\frac{\dot{Q}_{y+\Delta y} - \dot{Q}_y}{\Delta y} - \frac{1}{\Delta x \Delta y}\frac{\dot{Q}_{z+\Delta z} - \dot{Q}_z}{\Delta z} + \dot{g} = \rho C \frac{T_{t+\Delta t} - T_t}{\Delta t} \quad \textbf{(2-37)}$$

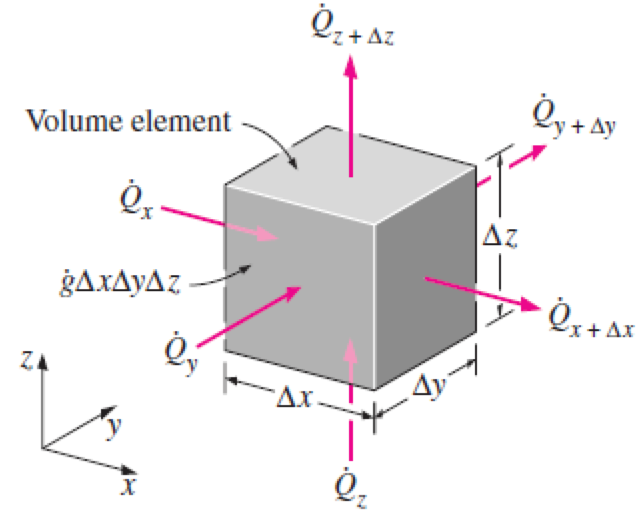


FIGURE 2–21
Three-dimensional heat conduction through a rectangular volume element.

Noting that the heat transfer areas of the element for heat conduction in the x, y, and z directions are $A_x = \Delta y\Delta z$, $A_y = \Delta x\Delta z$, and $A_z = \Delta x\Delta y$, respectively, and taking the limit as Δx, Δy, Δz and $\Delta t \to 0$ yields

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t} \qquad \text{(2-38)}$$

since, from the definition of the derivative and Fourier's law of heat conduction,

$$\lim_{\Delta x\to 0}\frac{1}{\Delta y\Delta z}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} = \frac{1}{\Delta y\Delta z}\frac{\partial Q_x}{\partial x} = \frac{1}{\Delta y\Delta z}\frac{\partial}{\partial x}\left(-k\Delta y\Delta z\frac{\partial T}{\partial x}\right) = -\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)$$

$$\lim_{\Delta y\to 0}\frac{1}{\Delta x\Delta z}\frac{\dot{Q}_{y+\Delta y} - \dot{Q}_y}{\Delta y} = \frac{1}{\Delta x\Delta z}\frac{\partial Q_y}{\partial y} = \frac{1}{\Delta x\Delta z}\frac{\partial}{\partial y}\left(-k\Delta x\Delta z\frac{\partial T}{\partial y}\right) = -\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$$

$$\lim_{\Delta z\to 0}\frac{1}{\Delta x\Delta y}\frac{\dot{Q}_{z+\Delta z} - \dot{Q}_z}{\Delta z} = \frac{1}{\Delta x\Delta y}\frac{\partial Q_z}{\partial z} = \frac{1}{\Delta x\Delta y}\frac{\partial}{\partial z}\left(-k\Delta x\Delta y\frac{\partial T}{\partial z}\right) = -\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

Equation 2–38 is the general heat conduction equation in rectangular coordinates. In the case of constant thermal conductivity, it reduces to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{(2-39)}$$

where the property $\alpha = k/\rho C$ is again the *thermal diffusivity* of the material.

Equation 2–39 is known as the **Fourier-Biot equation,** and it reduces to these forms under specified conditions:

(1) *Steady-state:* (called the **Poisson equation**)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = 0 \qquad \textbf{(2-40)}$$

(2) *Transient, no heat generation:* (called the **diffusion equation**)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \textbf{(2-41)}$$

(3) *Steady-state, no heat generation:* (called the **Laplace equation**)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad \textbf{(2-42)}$$

Note that in the special case of one-dimensional heat transfer in the x-direction, the derivatives with respect to y and z drop out and the equations above reduce to the ones developed in the previous section for a plane wall (Fig. 2–22).

Cylindrical

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad \text{and} \qquad z = z$$

After lengthy manipulations, we obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(kr\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t} \qquad \text{(2-43)}$$

Spherical

$$x = r\cos\phi\sin\theta, \qquad y = r\sin\phi\sin\theta, \qquad \text{and} \qquad z = \cos\theta$$

Again after lengthy manipulations, we obtain

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(k\frac{\partial T}{\partial\phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(k\sin\theta\frac{\partial T}{\partial\theta}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t} \tag{2-44}$$

EXAMPLE 2–6 Heat Conduction in a Short Cylinder

A short cylindrical metal billet of radius R and height h is heated in an oven to a temperature of 600°F throughout and is then taken out of the oven and allowed to cool in ambient air at $T_\infty = 65°F$ by convection and radiation. Assuming the billet is cooled uniformly from all outer surfaces and the variation of the thermal conductivity of the material with temperature is negligible, obtain the differential equation that describes the variation of the temperature in the billet during this cooling process.

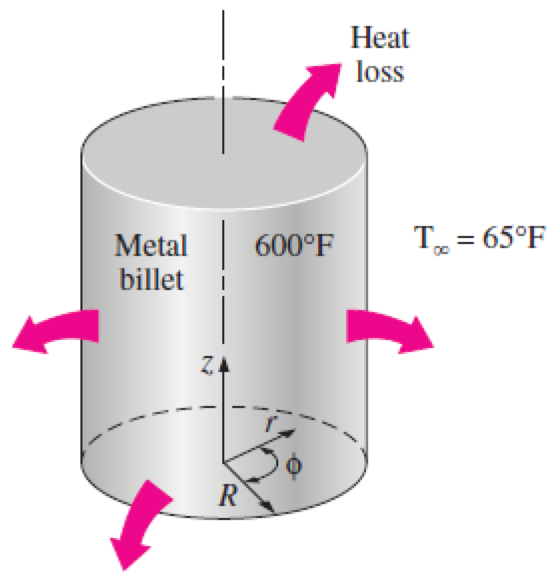


FIGURE 2–25
Schematic for Example 2–6.

SOLUTION The billet shown in Figure 2–25 is initially at a uniform temperature and is cooled uniformly from the top and bottom surfaces in the *z*-direction as well as the lateral surface in the radial *r*-direction. Also, the temperature at any point in the ball will change with time during cooling. Therefore, this is a two-dimensional transient heat conduction problem since the temperature within the billet will change with the radial and axial distances *r* and *z* and with time *t*. That is, $T = T(r, z, t)$.

The thermal conductivity is given to be constant, and there is no heat generation in the billet. Therefore, the differential equation that governs the variation of temperature in the billet in this case is obtained from Eq. 2–43 by setting the heat generation term and the derivatives with respect to ϕ equal to zero. We obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = \rho C\frac{\partial T}{\partial t}$$

In the case of constant thermal conductivity, it reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

which is the desired equation.